IDENTIFICATION OF STUDENT CHARACTERISTICS ASSOCIATED WITH ATTEMPTS AT SOLVING DEDUCTIVE PROBLEMS IN GEOMETRY

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Abstract

In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service primary teachers. The test which was carried out in an interview situation, was designed to examine seven geometric concepts. The Mayberry study has been replicated as a written test under Australian conditions. Analysis of the Australian results led to the identificaion of some problems with the Mayberry test items which had the potential to lead to incorrect assignment of a student's level of geometry. The analysis of these results was reported earlier (Lawrie, 1993). The analysis also led to the identification of several Level 4 Mayberry items which were seen as capable of assessing deductive skills. This paper analyses the responses to three of these items and discusses how these responses can be seen as indicators of a student's level of geometric reasoning.

Introduction

In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service primary teachers. The test, which was carried out in an interview situation, was designed to examine seven geometric concepts. In order to consider Mayberry's work in an Australian context, a detailed study of the geometric understanding of 60 first -year primary trainees was carried out at the University of New England. The study aimed, in part, to provide a written test based on the Mayberry interview schedule. Follow-up interviews were conducted with students to validate the levels of thinking as determined by the written test. When collating results in this study, some of the students' reasoning was not consistent with expectations according to the Mayberry items. On analysis of the results by concept and by Level, it was considered that certain aspects of the Mayberry items had the potential to lead to incorrect assignment of a student's level of understanding in geometry. The analysis of these inconsistencies was reported earlier (Lawrie, 1993). This paper continues the general analysis outlined previously, but focuses instead on questions in which there were no inconsistencies identified, i.e., questions which did give a reliable indication of the level of the students.

Background

While van Hiele (1986) has hypothesised five levels of insight, it is Levels 2, 3 and 4 which are of particular interest to this issue.

Level 2

In his thesis summary van Hiele (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.239) asserts "A pupil reaches the (second) level of thinking as soon as he can manipulate the known characteristics of a pattern that is familiar to him. For instance: if he is able to associate the name 'isosceles triangle' with a specific triangle, knowing that two of its sides are equal, and to draw the subsequent conclusion that the two corresponding angles are equal". Dina van Hiele-Geldof (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.191) identifies the main classroom features, "The goal of the (second) learning situations is to let the pupils experience the *aspect of geometry* in an empirical way. The pupils acquire visual geometric structures". Mayberry (1981, p.18) states "At level (2), properties are distinguished, but not organized. The student discovers necessary conditions, but the role of sufficient conditions is not perceived". In behavioural terms, Mayberry (p. 48) designed her questions to determine whether a student on this level could "recognize and name properties of geometric figures".

Level 3

Concerning a student's ability to reason with Level 3 skills, van Hiele explains (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.239) "As soon as he learns to manipulate the interrelatedness of the characteristic of geometric pattern he will have reached the (third) level of thinking, e.g. if, on the strength of general congruence theorems, he is able to deduce the equality of angles or linear segments of specific figures". Concerning the teacher's guidance of the students from Level 2 to Level 3, Dina van Hiele-Geldof explains, "By letting the pupils analyze at their level, an ordering of certain relations evolves. Known relations can be a consequence of other known relations and new relations can be discovered from known relations. The goal of these learning situations is to bring pupils from empiricism to the *essence of geometry.* Through this analysis it becomes possible for pupils to expand their visual geometric structures into structures that belong to the second level of thinking" (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.191). Mayberry (1981, p.18) suggests that "a student with level (2) skills understands that properties are organized, implications and class inclusions are perceived, but the significance of deduction as a whole is not understood. However, at Level 3, one property may be deduced from another, a proof may be followed but not yet be constructed, and the student can understand sufficient conditions in a definition". In behavioural terms Mayberry (p. 48) continues "a student should

1) Give definitions (since necessary and sufficient conditions are not understood, a definition may include superfluous conditions);

2) Recognise and name relationships;

3) Recognise class inclusions and implications."

Level 4

Van Hiele explains (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.240) that a student "will reach the (fourth) level of thinking when he starts manipulating the intrinsic characteristics of relations". More recently van Hiele (1986, p.44) suggests "A (fourth) level must be connected with the possibility of comparing, transposing and operating with relations". Dina van Hiele-Geldof explains (1957, in Fuys, Geddes, and Tischler, eds., 1984, p.192) that "as a result of the learning situation in which the pupils analyzed the theorems for their correctness, the pupils ascend to the (fourth) level of thinking". Concerning Level 4, Mayberry (1981, p.18) declares that "deduction is understood as is the role of necessary and sufficient conditions and a proof can be constructed according to the rules of logic. At this level the pupil can distinguish between a proposition and its converse". Behaviourally (p.49) "a student should

1) Supply reasons for steps in a proof;
2) Construct a proof."

The Study

The analysis of Mayberry's items has led to the identification of several Level 4 items which are capable of detecting deductive skills. None of these items suffered the disadvantage of including important prompts which influenced some of the other items. Because of space limitations only three items are to be looked at, Items 45, 47 and 50. These three items examine the concepts square, right triangle and isosceles triangle. This paper analyses the various written responses given by students and where appropriate, the interview data, and discusses how the responses can be seen as indicators of a student's level of geometric reasoning.

Results and Discussion

Item 45 (concept square)

ABCD is a four sided figure. Suppose we know that opposite sides are parallel. What are the fewest facts necessary to prove that ABCD is a square?

This item requires students to display the Level 4 skill of understanding necessary and sufficient conditions, i.e., to identify that a pair of adjacent sides must be equal and that one angle be a right angle. In addition, this item allows students to demonstrate their knowledge of other level skills. For example, a student displaying Level 3 skills (but not yet mastering Level 4) will include superfluous conditions, whereas a Level 2 response may include a list of all known properties or focus on a single familiar property.

Of the sixty-one students attempting this item, the best three responses could be considered transitional. The students understood the notion of necessary and sufficient conditions. However, they were not able to completely control all the elements. For example, S41 gave a typical transitional response, "AB = BC. 90° angles." This response provided too much information on angles. During the interview he provided an equivalent response, "All sides are equal in length. One angle equals 90°." Here, while be mentioned a single angle, he added more information than was needed about the sides. Prompting did not induce S41 to reduce both side and angle properties to a minimum. However it did produce the qualification that "without the right-angle the figure would be a rhombus".

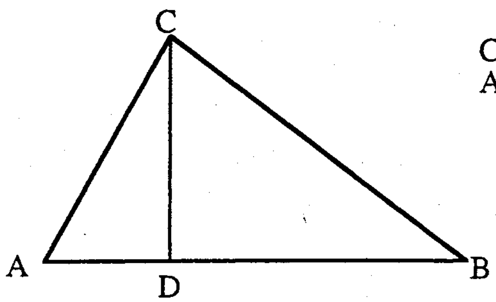
Level 3 responses were provided by seven students (11%). One student S14 demonstrated in her verbal responses a confidence with Level 3 skills together with an awareness that a deeper response was required for Item 45. However, she also could not produce that response. Her answer to this item in part was "I'm just trying to work out the difference between a rhombus and a square...do diagonals bisect at 90°?...all sides equal and diagonals bisect at 90°...a square is symmetrical...so is a rhombus." The student was then prompted with 'Can you do with less?' Her response was "You need to know all sides are equal, 90° angles and a rhombus isn't" thus demonstrating understanding of symmetry and class relations, but unable to extend her thinking to the requirement of necessary and sufficient conditions.

Fifty-one students (84%) failed to meet criteria for a level greater than Level 2 on this item. A response typical of this group is S 53's response "All sides and equal, all angles are right angles." S 53 did not indicate Level 3 skills in any of her responses for the concept square confirming that this is a Level 2 listing of all known properties and not an inability to minimise necessary conditions (Level 3).

Two other students both of whom displayed some knowledge of properties and neither meeting the Mayberry criteria for Level 2, gave interesting responses. S01 is a mature age student who has not studied any mathematics since the age of fourteen years. In her response "that all sides are the same length" she has focused on a single property. Although S01 attempted most questions for Levels 1, 2, and 3, at no time in her paper did S01 refer to any properties other than the properties of the sides of figures. S01 failed to meet Level 2 criteria for three of the four concepts on which she was tested. The second student, S37 displayed in her responses a similar ability to S01. She tended to give side properties only, but in contrast, quantified the properties whenever possible. Her response to Item 45 was "A = 2 cm, B = 2 cm, C = 2 cm, D = 2 cm."

The above responses to Item 45, when analysed in conjunction with each student's response to other questions, indicate clearly at what van Hiele level a student is working. It can therefore be seen that not only is Item 45 capable of assessing a student's Level 4 skills but also of indicating at what level students who have not reached Level 4 reasoning are functioning.

Item 47 (concept right triangle)



CD is perpendicular to AB
Angle C is a right angle.

If you measure $\angle ACD$ and $\angle B$, you will find that they have the same measure.
Would this equality be true for all right triangles? Why or why not?

To make the statement 'the angles will always be equal, each being the complement of the same angle' requires that a student is able to 1) solve a deductive problem, and 2) generalise from the above ΔABC to any right triangle. This item, while defining whether or not a student is operating at van Hiele Level 4, is unlike Item 45 in that it is difficult to differentiate clearly between other levels for students who have not yet developed deductive skills.

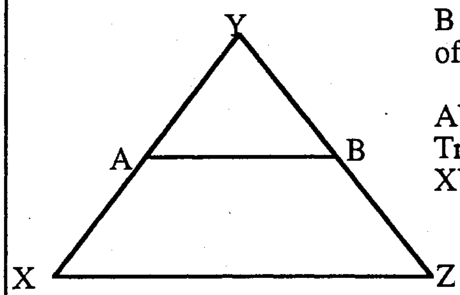
Mayberry included three items in testing for Level 4 deductive skills in the concept right triangle. Of the thirty-one students who were tested for the concept, only one student, S29 reached criteria (she was correct for all three questions). Her response reads "Yes, $\angle A$ and $\angle B$ must equal 90° in ΔABC. $\angle A$ and $\angle C$ must equal 90° in ΔADC. Therefore $\angle B = \angle C$". While this response is not expressed in the general terms above, the essence of the proof is there. S29 demonstrated overall deductive skills and hence can be considered as meeting Level 4 criteria skills, albeit in a limited sense.

A Level 3 attempt was made by S06. (It is interesting that S06 met Level 3 criteria for all concepts, but although attempting all Level 4 questions, nowhere gave any indication of deductive skills.) Her response to Item 47 "Yes, because the angles are on a line (180°) and it wouldn't matter about the length." While this answer is not particularly satisfactory, the student has tried to consider geometric relationships which are related to the figure as is presented and has tried to be consistent within that context.

Many of the responses indicate that students were working at Level 2. Some of the students achieved this by quantifying the problem, i.e., reducing the problem to a single aspect on which they could focus. S02 , S17 and S18 allocated 45° to all non-right angles in the diagram, thus *showing* the specified angles to be equal. These three students were consistent in their method, allocating 45° to the non-right angles for every question on the concept right triangle in the test. None of the three students met criteria for Level 3 in any concept. In contrast to the above four students, S05 focused on the sides of the triangle, replying 'No, it would depend on side lengths.' S21 also displayed a lack of awareness of the purpose of the question in responding 'Yes, because opposite angles are equal to each other'.

Thus it can be seen that Item 47 not only is capable of discerning whether or not a student has mastered deductive skills, it can also give an indication of a student's van Hiele level of reasoning.

Item 50 (concept isosceles triangle)



AB is the line segment with A and B the midpoints of the equal sides of the isosceles triangle XYZ.

AY is equal to BY.
Triangle AYB is similar to triangle XYZ.

So angle A is equal to angle X and AB is parallel to XZ.
What have we proved?

This item requires a student to understand the essence of deduction to be able to follow through the above deductive steps, $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e$, leading to the conclusion that the line joining the mid-points of two sides of an isosceles triangle is parallel to the third side. Mayberry included three items of this type in her interviews. The Mayberry students experienced difficulties with these items, one student only giving a correct response to Item 51. The Australian students found the items equally difficult. S41's response for Item 50 was the only response which demonstrated deductive understanding for any of the three items. S41's response "that in an isosceles triangle any line segment parallel to the base will form a similar isosceles triangle", while not giving the complete answer, displays a degree of deductive understanding. In his interview, S41 provided further insight into his deductive understanding by repeating his solution in a slightly different way. "We can move AB anywhere there and maintain similar triangles as long as the ratios are the same. If the ratios are not the same, AB is no longer parallel to XZ and the triangles are not similar". In both cases he demonstrated his ability to manipulate geometric relationships.

Six (20%) other responses were of the type a→c, i.e., they responded that the given implied a statement half-way through the proof. This type of answer appears to be an early attempt at deduction. An example is S36's response that "The triangles are similar, ∠A = ∠X, ∠B = ∠Z". A further eight students (27%) identified a single piece of missing information, for example, "ΔAYB is isosceles". Some of the students responding in this manner displayed Level 3 skills in several concepts, indicating that this type of response could also be the deduction of one property from another. However, not all students giving this answer demonstrated Level 3 ability elsewhere.

One student, S33 who gave the response, "ΔAYB is isosceles" in her written paper met Level 3 criteria for this concept. When interviewed her solution "∠A = ∠X, AB is parallel to XZ" was of the type a→c, demonstrating early deductive skills. When probed she concentrated on angle relations and was satisfied that as long as AB remained parallel to XZ she could still find equal corresponding angles. Another student who was interviewed, S31, showed similar overall reasoning ability to S33. S31 had not seen the item before the interview. His initial response was "We've proved that both triangles AYB and XYZ are similar to each other". He then continued, "The question is asking you to copy information onto the diagram and think the processes through...They could also be asking what other angles are equal. ∠ABY = ∠YZX....to find angle sizes you just take the number you know from 180°". As with Item 45, again a Level 3 student is aware that the question requires a deeper response and also that he cannot produce that answer. This awareness of the depth of the question was demonstrated in several students' responses both in the written test as well as in interviews. By contrast, in six responses (20%) a single step in the stated proof was selected. As no deductive skills are demonstrated in such a response, it is suggestive of a Level 2 answer. S38's response, "ΔAYB is similar to ΔXYZ" was of this type.

Analysis of the responses above underlines the reliability of this type of item as an instrument for assessing whether or not a student is capable of understanding the essence of deduction. Again, as in Item 47, responses to this item can give an indication of a student's van Hiele level of reasoning.

Conclusion

On analysing the Level 4 responses in the Australian study, it was found that some of the Mayberry items led to inconsistencies in student assessment. These items suffered the disadvantage of including significant prompts which could limit their ability to test deductive skills. Such items allow a student to manipulate the interrelatedness of geometric pattern, i.e., to demonstrate Level 3 skills. Van Hiele's descriptions of Level 3 skills above support this distinction between items which can and cannot test deductive skills. Mayberry's behavioural terms for Level 4 may not be sufficiently concise. The Australian analysis also identified several Mayberry Level 4 items which did have the potential to detect deductive skills. None of these items suffered the disadvantage of including important prompts. This paper has analysed students' responses for three of these items and discussed how the responses can be seen as indicators of a student's level of geometric reasoning. Item 45 is not only capable of identifying the understanding of necessary and sufficient conditions, but is also capable of indicating at what level students who have not reached Level 2 reasoning are functioning. Item 47 requires a student to identify a Level 3 concept (angle sum = 180°), then use it as a tool to solve the deductive problem. Analysis of the responses to Item 50 underlines the reliability of the item as an instrument for assessing whether or not a student understands the essence of deduction. Items 47 and 50 not only are capable of discerning Level 4 skills, both can also give an indication of a student's van Hiele level of geometric reasoning

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